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Partition Theorems for Subspaces of Vector Spaces

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The principal result of this paper provides a nearly complete answer to the following question. For which cardinal numbers $\mathfrak{k}$, $\mathfrak{m}$, $\mathfrak{n}$, $\mathfrak{q}$, and $\mathfrak{r}$ is it true that whenever the $\mathfrak{k}$-dimensional subspaces of an $\mathfrak{n}$-dimensional vector space V over a field of $\mathfrak{q}$ elements are partitioned into $\mathfrak{r}$ classes, there must be some $\mathfrak{m}$-dimensional subspace of V, all of whose $\mathfrak{k}$-dimensional subspaces lie in the same class? This question is answered completely if $\mathfrak{r} < \aleph_0$. The contributions of this paper are in the form of negative answers, since it turns out that all affirmative answers (which we have) were already known or easily deducible from known results.

1. INTRODUCTION

In order to simplify the discussion we introduce the following notation.

1.1. *Notation.* (a) Let V be a vector space and let $\mathfrak{k}$ be a cardinal. $\Gamma(\mathfrak{k}, V) = \{U: U \text{ is a } \mathfrak{k}\text{-dimensional subspace of } V\}$.

(b) Let $\mathfrak{k}$, $\mathfrak{m}$, $\mathfrak{n}$, $\mathfrak{q}$, and $\mathfrak{r}$ be cardinals. Then $\Psi(\mathfrak{k}, \mathfrak{m}, \mathfrak{n}, \mathfrak{q}, \mathfrak{r})$ represents the statement: "Whenever V is an $\mathfrak{n}$-dimensional vector space over a field with $\mathfrak{q}$ elements and $\Gamma(\mathfrak{k}, V) = \bigcup_{\sigma<\mathfrak{r}} A_\sigma$, there are some U in $\Gamma(\mathfrak{m}, V)$ and some $\sigma < \mathfrak{r}$ such that $\Gamma(\mathfrak{k}, U) \subseteq A_\sigma$.

The object of this paper is to determine the cardinals for which $\Psi(\mathfrak{k}, \mathfrak{m}, \mathfrak{n}, \mathfrak{q}, \mathfrak{r})$ holds. In [2, Corollary 2] the following conjecture of Rota was proved.

1.2. THEOREM (Graham, Leeb, and Rothschild). *Let* $\mathfrak{k}$, $\mathfrak{m}$, *and* $\mathfrak{r}$ *be positive integers and let* $\mathfrak{q} = 2^{\mathfrak{p}}$ *for some prime* $\mathfrak{p}$. *Then there is a smallest positive integer* $N(\mathfrak{k}, \mathfrak{m}, \mathfrak{q}, \mathfrak{r})$ *such that* $\Psi(\mathfrak{k}, \mathfrak{m}, \mathfrak{n}, \mathfrak{q}, \mathfrak{r})$ *holds whenever* $\mathfrak{n} > N(\mathfrak{k}, \mathfrak{m}, \mathfrak{q}, \mathfrak{r})$.

It was proved in [4, Corollary 3.5] that $\Psi(1, \aleph_0, \mathfrak{n}, 2, \mathfrak{r})$ holds whenever

$\mathfrak{r} < \aleph_0$ and $\mathfrak{n} \geqslant \aleph_0$. In addition to these two results there are of course several trivial cases in which $\Psi(\mathfrak{k}, \mathfrak{m}, \mathfrak{n}, \mathfrak{q}, \mathfrak{r})$ holds, (for example if $\mathfrak{q} = 6$, in which case there is no field of cardinality $\mathfrak{q}$, or $\mathfrak{k} = 0$ or $\mathfrak{r} = 1$). It will be shown in Section 3 that, for $\mathfrak{r}$ finite, these are the only nontrivial cases for which $\Psi(\mathfrak{k}, \mathfrak{m}, \mathfrak{n}, \mathfrak{q}, \mathfrak{r})$ holds.

If $\mathfrak{r} \geqslant \aleph_0$ there is another class of cases for which $\Psi(\mathfrak{k}, \mathfrak{m}, \mathfrak{n}, \mathfrak{q}, \mathfrak{r})$ holds, provided one assumes the generalized continuum hypothesis. That is, if $\mathfrak{m} < \aleph_0$ and $\sigma \geqslant 2^{\mathfrak{m}-1}$, then $\Psi(1, \mathfrak{m}, \aleph_{\alpha+\sigma}, 2, \aleph_\alpha)$ holds. This result is an easy consequence of a result of Erdös and Rado [1] and of Kurepa [5]. The proof of this result, as well as the proofs of the negative results, will be found in Section 2. The main theorems are in Section 3.

Section 4 contains some remarks about related problems and some open questions.

2. Development of Results

Throughout this paper lowercase Gothic letters will represent cardinal numbers. (A cardinal is interpreted as the first ordinal of an equipotence class.) Lowercase Greek letters will represent ordinals. Thus the statement $\sigma < \mathfrak{n}$ means that $\mathfrak{n}$ is a cardinal, σ is an ordinal, and σ is less than $\mathfrak{n}$.

A hypothesized $\mathfrak{n}$-dimensional vector space V over a field F will be viewed as the set of $\mathfrak{n}$-tuples (that is, functions from the ordinal $\mathfrak{n}$) of elements of F, all but finitely many of whose coordinates are 0. In particular, when $\sigma < \mathfrak{n}$ and $v \in V$, the σth coordinate of v is denoted by $v(\sigma)$. Also the symbol "0" will be used interchangeably to denote the zero of F, the zero of V, and the ordinal zero.

2.1. Definition. Let V be an $\mathfrak{n}$-dimensional vector space and let $U \subseteq V$.

(a) $S(U) = \{\sigma < \mathfrak{n}$: there is some $v \in U$ such that $v(\sigma) \neq 0\}$;

(b) $OS(U) = \mathrm{Ord}(S(U))$, (that is, $OS(U)$ is the ordinal whose order type is the same as that which $S(U)$ inherits from $\mathfrak{n}$);

(c) $\bar{U}$ is the order isomorphism from $OS(U)$ onto $S(U)$;

(d) $s(U) = |S(U)|$;

(e) for u in V, $OI(u, U)$ is the $OS(U)$-tuple whose σth coordinate is $u(\bar{U}(\sigma))$.

In the above definitions S, OS, and OI are intended to represent "support," "ordered support," and "ordered image" respectively. If $v \in V$ we shall write $S(v)$ for $S(\{v\})$ and $s(v)$ for $s(\{v\})$. We shall also write $\langle A \rangle$ and $\langle a_1, a_2, \ldots, a_t \rangle$ for the vector spaces generated by A and $\{a_1, a_2, \ldots, a_t\}$ respectively.

2.2. LEMMA. *Let* $\mathfrak{m} \geqslant \aleph_0$ *and let* $\mathfrak{n} \geqslant \mathfrak{m}$. *Let* F *be any field and let* V *be an* $\mathfrak{n}$*-dimensional vector space over* F. *Let* $U \in \Gamma(\mathfrak{m}, V)$. *Then there exists* $\{u_\sigma : \sigma < \mathfrak{m}\} \subseteq U \backslash \{0\}$ *such that* $S(u_\sigma) \cap S(u_\tau) = \varnothing$ *whenever* $\sigma \neq \tau$. *If* $\mathfrak{m} = \aleph_0$, *then* $\{u_\sigma : \sigma < \mathfrak{m}\}$ *may be chosen so that in addition* $\inf(S(u_\sigma)) < \inf(S(u_\tau))$ *and* $\sup(S(u_\sigma)) < \sup(S(u_\tau))$ *whenever* $\sigma < \tau < \mathfrak{m}$.

Proof. Let $u_0 \in U \backslash \{0\}$. Let $\alpha < \mathfrak{m}$ and suppose we have chosen $G = \{u_\sigma : \sigma < \alpha\} \subseteq U \backslash \{0\}$ with the property that $S(u_\sigma) \cap S(u_\tau) = \varnothing$ whenever $\sigma < \tau < \alpha$. Let W be a set of $\mathfrak{m}$ linearly independent elements of U. Since $s(G) < \mathfrak{m}$, the set of $OS(G)$-tuples over F is a vector space of dimension less than $\mathfrak{m}$. Thus there are some $\mathfrak{t} < \aleph_0$, $\{w_\mathfrak{i}\}_{\mathfrak{i}=1}^{\mathfrak{t}} \subseteq W$, and $\{a_\mathfrak{i}\}_{\mathfrak{i}=1}^{\mathfrak{t}} \subseteq F$, not all 0, such that $\sum_{\mathfrak{i}=1}^{\mathfrak{t}} a_\mathfrak{i} OI(w_\mathfrak{i}, G) = 0$. But that is the same as saying that $\sum_{\mathfrak{i}=1}^{\mathfrak{t}} a_\mathfrak{i} w_\mathfrak{i}$ is zero on all of $S(G)$. Consequently we may let $u_\alpha = \sum_{\mathfrak{i}=1}^{\mathfrak{t}} a_\mathfrak{i} w_\mathfrak{i}$. Since W was a linearly independent set, $u_\alpha \neq 0$.

To see the second statement note that $\{u_\sigma : \sigma < \omega_0\}$ may be reordered so that, if $\sigma < \tau < \omega_0$, then the order type of the interval from $\inf(S(u_\sigma))$ to $\sup(S(u_\sigma))$ is no larger than the order type of the interval from $\inf(S(u_\tau))$ to $\sup(S(u_\tau))$. (In fact this can be done for any $\mathfrak{m}$.)

Now let v_0 be the element of $\{u_\sigma : \sigma < \omega_0\}$ whose support begins first, and inductively, if $v_\tau = u_\delta$, let $v_{\tau+1}$ be the element of $\{u_\sigma : \delta < \sigma < \omega_0\}$ whose support begins first. Then one has directly that $\inf(S(v_\tau)) < \inf(S(v_{\tau+1}))$. (The strict inequality holds since the supports are pairwise disjoint.) Also, since the interval from first to last support element in $v_{\tau+1}$ is at least as long as in v_τ, one concludes that $\sup(S(v_\tau)) < \sup(S(v_{\tau+1}))$ as desired.

It may be remarked that the second statement of Lemma 2.2 holds whenever $\mathfrak{m}$ is any regular cardinal.

We are now ready to develop the negative answers to the question of the validity of $\Psi(\mathfrak{k}, \mathfrak{m}, \mathfrak{n}, \mathfrak{q}, \mathfrak{r})$. In all of the following we shall assume that the conclusions about $\Psi(\mathfrak{k}, \mathfrak{m}, \mathfrak{n}, \mathfrak{q}, \mathfrak{r})$ are not vacuous. Thus, *throughout this paper it will be assumed that* $\mathfrak{k} \leqslant \mathfrak{m} \leqslant \mathfrak{n}$, $\mathfrak{r} > 0$, *and* $\mathfrak{q}$ *is either infinite or a positive power of a prime.*

2.3. LEMMA. *Let* $\mathfrak{k} \geqslant 2$ *and* $\mathfrak{m} \geqslant \aleph_0$. *Then* $\Psi(\mathfrak{k}, \mathfrak{m}, \mathfrak{n}, \mathfrak{q}, 2)$ *fails for every* (*permissible*) $\mathfrak{n}$ *and* $\mathfrak{q}$.

Proof. Let V be an $\mathfrak{n}$-dimensional vector space over a field with $\mathfrak{q}$ elements. For each W in $\Gamma(\mathfrak{k}, V)$ choose w_0 and w_1 in W such that $s(w_0) = \min\{s(w) : w \in W\}$ and $s(w_1) = \min\{s(w) : \{w_0, w\}$ is a linearly independent subset of $W\}$. If $s(w_0) \mid s(w_1)$ let $W \in A_0$ and otherwise let $W \in A_1$.

Now let $U \in \Gamma(\mathfrak{m}, V)$ and let $\{u_\sigma : \sigma < \mathfrak{m}\}$ be as guaranteed by Lemma 2.2. Since $\mathfrak{m} \geqslant \aleph_0$, there exists a finite $G \subseteq \{u_\sigma : 0 < \sigma < \mathfrak{m}\}$ such that

$s(u_0) \mid \sum \{s(u_\sigma)\colon u_\sigma \in G\}$. But, since $S(u_\sigma) \cap S(u_\tau) = \varnothing$ when $\sigma \neq \tau$, one has $s(u_0) \mid s(v_1)$ where $v_1 = \sum G$. Let $v_0 = u_0$. Now for each σ, with $1 < \sigma < \mathfrak{k}$, let $v_\sigma \in U$ such that $s(v_1) < s(v_\sigma)$ and distinct v_σ's share no nonzero coordinates. (This is done by combining sufficiently many members of $\{u_\sigma\colon \sigma < \mathfrak{m}\}$.) Let $W = \langle\{v_\sigma\colon \sigma < \mathfrak{k}\}\rangle$. Then $w_0 = v_0$ and $w_1 = v_1$ so that $W \in A_0$.

In a similar fashion one easily combines the members of $\{u_\sigma\colon \sigma < \mathfrak{m}\}$ to obtain an element of $\Gamma(\mathfrak{k}, U) \cap A_1$.

2.4. LEMMA. *Let* $\mathfrak{m} \geqslant \aleph_0$ *and let* $\mathfrak{q} > 2$. *Then* $\Psi(1, \mathfrak{m}, \mathfrak{n}, \mathfrak{q}, 2)$ *fails for every* $\mathfrak{n}$.

Proof. It suffices to show that $\Psi(1, \aleph_0, \mathfrak{n}, \mathfrak{q}, 2)$ fails. (For, if $\Gamma(1, U) \subseteq A_\sigma$ and $U' \in \Gamma(\aleph_0, U)$, then $\Gamma(1, U') \subseteq A_\sigma$.)

Let V be an $\mathfrak{n}$-dimensional vector space over a field F with $\mathfrak{q}$ elements. For each $W \in \Gamma(1, V)$ let $w \in W\backslash\{0\}$. Let $\tau = \inf(S(w))$ and let $\gamma = \sup(S(w))$. If $w(\tau) = w(\gamma)$ let $W \in A_0$. Otherwise let $W \in A_1$. Note that this assignment does not depend on the choice of w in W.

Now, let $U \in \Gamma(\aleph_0, V)$ and let $\{u_\sigma\colon \sigma < \aleph_0\}$ be as guaranteed by Lemma 2.2. Let $\tau = \inf(S(u_0))$ and let $\gamma = \sup(S(u_1))$. (Recall that $\tau < \inf(S(u_1))$ and $\gamma > \sup(S(u_0))$.) Let $a = u_0(\tau)$ and let $b = u_1(\gamma)$. Let $w_0 = u_0 + ab^{-1}u_1$. Then $w_0(\tau) = w_0(\gamma)$ so $\langle w_0\rangle \in A_0$. If $a \neq b$ let $w_1 = u_0 + u_1$. If $a = b$ let $c \in F\backslash\{0, 1\}$, and let $w_1 = u_0 + cu_1$. In either case $w_1(\tau) \neq w_1(\gamma)$ so $\langle w_1\rangle \in A_1$.

The authors are grateful to P. Erdös for bringing the proof of the following lemma to their attention.

2.5. LEMMA (Erdös). *Let* $\mathfrak{m} > \aleph_0$. *Then* $\Psi(1, \mathfrak{m}, \mathfrak{n}, 2, 2)$ *fails for all* $\mathfrak{n}$.

Proof. Let V be an $\mathfrak{n}$-dimensional vector space over $GF(2)$. Let $W \in \Gamma(1, V)$. If $s(W) = 2^{2\mathfrak{t}}(2\mathfrak{i} + 1)$ for some $\mathfrak{t}$ and $\mathfrak{i}$ let $W \in A_0$. Otherwise let $W \in A_1$.

Let $U \in \Gamma(\mathfrak{m}, V)$ and let $\{u_\sigma\colon \sigma < \mathfrak{m}\}$ be as given by Lemma 2.2. Since $\mathfrak{m} > \aleph_0$ there exist distinct σ and τ such that $s(u_\sigma) = s(u_\tau)$. Then $\langle u_\sigma\rangle \in A_0$ if and only if $\langle u_\sigma + u_\tau\rangle \in A_1$.

The following lemma is easily established by familiar cardinality arguments.

2.6. LEMMA. *Let* $\mathfrak{k} < \aleph_0$ *and let* $\mathfrak{k} < \mathfrak{n} \leqslant \mathfrak{q}$ *with* $\mathfrak{q} \geqslant \aleph_0$. *Let V be an* $\mathfrak{n}$*-dimensional vector space over a field with* $\mathfrak{q}$ *elements. Then* $|\Gamma(\mathfrak{k}, V)| = \mathfrak{q}$.

2.7. LEMMA. *Let* $0 < \mathfrak{k} < \mathfrak{m} < \aleph_0$ *and let* $\mathfrak{q} \geqslant \aleph_0$. *Then* $\Psi(\mathfrak{k}, \mathfrak{m}, \mathfrak{n}, \mathfrak{q}, 2)$ *fails for all* $\mathfrak{n}$.

Proof. Let V be an $\mathfrak{n}$-dimensional vector space over a field with $\mathfrak{q}$ elements. We shall assume first that $\mathfrak{n} \leqslant \mathfrak{q}$. In this case, by Lemma 2.6, $|\Gamma(\mathfrak{m}, V)| = \mathfrak{q}$ (unless, of course, $\mathfrak{n} = \mathfrak{m}$, in which case the conclusion of the lemma is trivial since $\mathfrak{m} < \aleph_0$). Also by Lemma 2.6, whenever $U \in \Gamma(\mathfrak{m}, V)$ one has $|\Gamma(\mathfrak{k}, U)| = \mathfrak{q}$. Consequently we may write $\Gamma(\mathfrak{m}, V) = \{U_\sigma : \sigma < \mathfrak{q}\}$ and inductively assign one member of $\Gamma(\mathfrak{k}, U_\sigma)$ to each of A_0 and A_1.

Thus we may assume that $\mathfrak{n} > \mathfrak{q}$. For each $U \in \Gamma(\mathfrak{m}, V)$ and each W in $\Gamma(\mathfrak{k}, V)$, let $\Delta(W, U) = \{T \in \Gamma(\mathfrak{k}, U) : \{OI(w, W) : w \in W\} = \{OI(t, T) : t \in T\}$. (Intuitively $\Delta(W, U)$ is the set of $\mathfrak{k}$-dimensional subspaces of U which, when viewed on their support, look like W.) We claim that, for each U in $\Gamma(\mathfrak{m}, V)$ and each W in $\Gamma(\mathfrak{k}, V)$, $|\Delta(W, U)| < \aleph_0$. For if $S(T) = S(R)$ and $\{OI(t, T) : t \in T\} = \{OI(t, R) : t \in R\}$ then indeed $T = R$. Thus, if $S(T) = S(R)$ and $\{T, R\} \subseteq \Delta(W, U)$, then $T = R$. But, if $T \in \Delta(W, U)$, then $S(T) \subseteq S(U)$. Since there are only finitely many subsets of $S(U)$ the claim is established.

Note that $|\{U \in \Gamma(\mathfrak{m}, V) : S(U) \subseteq \omega_0\}| = \mathfrak{q}$, by Lemma 2.6. Also, $|\Gamma(\mathfrak{m}, V)| = \mathfrak{n}$ and $\mathfrak{n} > \mathfrak{q}$. Thus we may write $\Gamma(\mathfrak{m}, V) = \{U_\sigma : \sigma < \mathfrak{n}\}$ with the stipulation that $\sigma < \mathfrak{q}$ if and only if $S(U_\sigma) \subseteq \omega_0$.

Now let $T \in \Gamma(\mathfrak{k}, U_0)$. Then $|\Delta(T, U_0)| < \aleph_0$ while $|\Gamma(\mathfrak{k}, U_0)| = \mathfrak{q}$. Thus there exists R in $\Gamma(\mathfrak{k}, U_0) \backslash \Delta(T, U_0)$. Also, trivially, $\Delta(T, U_0) \cap \Delta(R, U_0) = \varnothing$. Let $B_0 = \Delta(T, U_0)$ and let $C_0 = \Delta(R, U_0)$. Assume that for each $\gamma < \alpha$ we have chosen B_γ and C_γ satisfying the following inductive hypotheses: (1) $B_\gamma \cup C_\gamma \subseteq \Gamma(\mathfrak{k}, U_\gamma)$, $B_\gamma \neq \varnothing$, and $C_\gamma \neq \varnothing$; (2) $(\bigcup_{\tau \leqslant \gamma} B_\tau) \cap (\bigcup_{\tau \leqslant \gamma} C_\tau) = \varnothing$; (3) if $\tau \leqslant \gamma$; $T \in B_\tau$, and $R \in C_\tau$, then $\Delta(T, U_\gamma) \subseteq B_\gamma$ and $\Delta(R, U_\gamma) \subseteq C_\gamma$; (4) if $\gamma < \aleph_0$ then $|\bigcup_{\tau \leqslant \gamma} (B_\tau \cup C_\tau)| < \aleph_0$; and (5) if $\aleph_0 \leqslant \gamma < \mathfrak{q}$ then $|\bigcup_{\tau \leqslant \gamma} (B_\tau \cup C_\tau)| \leqslant |\gamma|$.

Each of these hypotheses is easily seen to hold at 0. (To see (3) note that if $W \in \Delta(T, U_0)$ then $\Delta(W, U_0) = \Delta(T, U_0)$.)

The induction proceeds by three cases. In the first we assume $\alpha < \aleph_0$. Let $B_\alpha' = \bigcup \{\Delta(T, U_\alpha) : T \in B_\gamma$ for some $\gamma < \alpha\}$ and let $C_\alpha' = \bigcup \{\Delta(T, U_\alpha) : T \in C_\gamma$ for some $\gamma < \alpha\}$. Then by hypothesis (4) and the fact that $|\Delta(T, U)| < \aleph_0$ for each T in $\Gamma(\mathfrak{k}, V)$, we have that $|B_\alpha' \cup C_\alpha'| < \aleph_0$. Also $|\Gamma(\mathfrak{k}, U_\alpha)| = \mathfrak{q}$ and $\mathfrak{q} \geqslant \aleph_0$ so there exists T in $\Gamma(\mathfrak{k}, U_\alpha) \backslash C_\alpha'$. Let $B_\alpha = B_\alpha' \cup \Delta(T, U_\alpha)$. Similarly, since $|\Delta(T, U_\alpha)| < \aleph_0$, there exists R in $\Gamma(\mathfrak{k}, U_\alpha) \backslash B_\alpha$. Let $C_\alpha = C_\alpha' \cup \Delta(R, U_\alpha)$. Each of the inductive hypotheses is easily seen to hold. (Hypothesis (2) requires a little checking but it is a straightforward exercise.)

For the second case of the induction assume that $\aleph_0 \leqslant \alpha < \mathfrak{q}$. (Of course, if $\mathfrak{q} = \aleph_0$, this case is vacuous.) As before, let $B_\alpha' = \bigcup \{\Delta(T, U_\alpha) : T \in B_\gamma$ for some $\gamma < \alpha\}$ and let $C_\alpha' = \bigcup \{\Delta(T, U_\alpha) : T \in C_\gamma$ for some $\gamma < \alpha\}$. Then, for each $\gamma < \alpha$, we have either $|B_\alpha \cup C_\alpha| < \aleph_0$, (if $\gamma < \aleph_0$), or $|B_\gamma \cup C_\gamma| \leqslant |\gamma|$, (if $\aleph_0 \leqslant \gamma$). In either case $|B_\gamma \cup C_\gamma| \leqslant |\alpha|$ so that

$|\bigcup_{\gamma<\alpha}(B_\gamma \cup C_\gamma)| \leqslant |\alpha|$. Consequently $|B_\alpha' \cup C_\alpha'| \leqslant |\alpha| \cdot \aleph_0 = |\alpha|$. We have here $\mathfrak{q} > |\alpha|$ so there exists T in $\Gamma(\mathfrak{k}, U_\alpha)\backslash C_\alpha'$ and, letting $B_\alpha = B_\alpha' \cup \Delta(T, U_\alpha)$, there exists R in $\Gamma(\mathfrak{k}, U_\alpha)\backslash B_\alpha$. Let $C_\alpha = C_\alpha' \cup \Delta(R, U_\alpha)$. Then, as above, all hypotheses are easily verified.

For the final case of the induction assume that $\alpha \geqslant \mathfrak{q}$. Let $B_\alpha = \bigcup\{\Delta(T, U_\alpha): T \in B_\gamma \text{ for some } \gamma < \alpha\}$ and let $C_\alpha = \bigcup\{\Delta(T, U_\alpha): T \in C_\gamma \text{ for some } \gamma < \alpha\}$. Then each of the hypotheses except possibly (1) is quickly verified.

Let $\{u_1, u_2, ..., u_\mathfrak{m}\}$ be a basis for U_α. Define, for $i \in \{1, 2, ..., \mathfrak{m}\}$, $v_i \in V$ by $v_i(\sigma) = OI(u_i, U_\alpha)(\sigma)$ if $\sigma < s(U_\sigma)$ and $v_i(\sigma) = 0$ otherwise. (It may help to note that, since $\mathfrak{m} < \aleph_0$, $s(U_\alpha) = OS(U_\alpha)$.) Let $U = \langle v_1, v_2, ..., v_\mathfrak{m}\rangle$. Then $S(U) \subseteq \omega_0$ so $U = U_\gamma$ for some $\gamma < \mathfrak{q}$. Consequently there exist T in B_γ and R in C_γ. Thus we have that $\Delta(T, U_\alpha) \neq \varnothing$ and $\Delta(R, U_\alpha) \neq \varnothing$ so $B_\alpha \neq \varnothing$ and $C_\alpha \neq \varnothing$. The induction is complete.

Let $A_0 = \bigcup_{\alpha<\mathfrak{n}} B_\alpha$ and let $A_1 = \Gamma(\mathfrak{k}, V)\backslash A_0$ (so that $A_1 \supseteq \bigcup_{\alpha<\mathfrak{n}} C_\alpha$). By hypothesis (1), $\Gamma(\mathfrak{k}, U_\alpha) \cap A_0 \neq \varnothing$ and $\Gamma(\mathfrak{k}, U_\alpha) \cap A_1 \neq \varnothing$ for each $\alpha < \mathfrak{n}$.

All results necessary to determine the validity of $\Psi(\mathfrak{k}, \mathfrak{m}, \mathfrak{n}, \mathfrak{q}, \mathfrak{r})$ when $\mathfrak{r} < \aleph_0$ have now been obtained. The remaining results partially describe the validity of $\Psi(\mathfrak{k}, \mathfrak{m}, \mathfrak{n}, \mathfrak{q}, \mathfrak{r})$ when $\mathfrak{r} \geqslant \aleph_0$.

2.8. LEMMA. *$\Psi(1, \aleph_0, \mathfrak{n}, 2, \aleph_0)$ fails for all $\mathfrak{n}$.*

Proof. Let V be an $\mathfrak{n}$-dimensional vector space over $GF(2)$. For each $\sigma < \omega_0$ let $A_\sigma = \{W \in \Gamma(1, V): s(W) = \sigma\}$. Let $U \in \Gamma(\aleph_0, V)$ and let $\{u_\sigma: \sigma < \omega_0\}$ be as guaranteed by Lemma 2.2. Then $s(\langle u_0\rangle) \neq s(\langle u_0 + u_1\rangle)$.

2.9. LEMMA. *Let $0 < \mathfrak{k} < \mathfrak{m} < \aleph_0$ and let $\mathfrak{q} < \aleph_0$. If $\mathfrak{q} > 2$ or $\mathfrak{k} \neq 1$ then $\Psi(\mathfrak{k}, \mathfrak{m}, \mathfrak{n}, \mathfrak{q}, \aleph_0)$ fails for all $\mathfrak{n}$.*

Proof. Let V be an $\mathfrak{n}$-dimensional vector space over a field F with $\mathfrak{q}$ elements. For each σ, with $\mathfrak{k} \leqslant \sigma < \omega_0$, let V_σ be the set of σ-tuples of elements of F. Let $\mathfrak{j}_\sigma = |\Gamma(\mathfrak{k}, V_\sigma)|$. Then $\mathfrak{j}_\sigma < \aleph_0$. Write $\Gamma(\mathfrak{k}, V_\sigma) = \{W(\sigma, \mathfrak{i}): \mathfrak{i} < \mathfrak{j}_\sigma\}$.

For each W in $\Gamma(\mathfrak{k}, V)$ let $W^* = \{OI(w, W): w \in W\}$. Then $W^* \in \Gamma(\mathfrak{k}, V_\sigma)$ where $\sigma = s(W)$. Now for each $\sigma < \omega_0$ and $\mathfrak{i} < \mathfrak{j}_\sigma$ let $A_{\sigma,\mathfrak{i}} = \{W \in \Gamma(\mathfrak{k}, V): W^* = W(\sigma, \mathfrak{i})\}$. (If $\mathfrak{n} < \aleph_0$ then $A_{\sigma,\mathfrak{i}} = \varnothing$ when $\sigma > \mathfrak{n}$.) Then $\Gamma(\mathfrak{k}, V) = \bigcup_{\sigma<\omega_0} \bigcup_{\mathfrak{i}<\mathfrak{j}_\sigma} A_{\sigma,\mathfrak{i}}$.

Suppose there exist $U \in \Gamma(\mathfrak{m}, V)$, $\sigma < \omega_0$, and $\mathfrak{i} < \mathfrak{j}_\sigma$ such that $\Gamma(\mathfrak{k}, U) \subseteq A_{\sigma,\mathfrak{i}}$. We may assume that $\mathfrak{m} = \mathfrak{k} + 1$ (for, if $U' \in \Gamma(\mathfrak{k} + 1, U)$ then $\Gamma(\mathfrak{k}, U') \subseteq \Gamma(\mathfrak{k}, U)$). Let $\{u_1, u_2, ..., u_\mathfrak{m}\}$ be a basis for U.

Write $S(U) = \{\alpha_1, \alpha_2, ..., \alpha_\tau\}$ where $\tau = s(U)$ and $\alpha_\mathfrak{t} < \alpha_{\mathfrak{t}+1}$. (That is $\alpha_\mathfrak{t} = \overline{U}(\mathfrak{t})$ where $\overline{U}$ is defined in 2.1(c).) We may assume that $u_1(\alpha_1) = 1$.

(Some u_t has $u_t(\alpha_1)$ nonzero and we may reorder and multiply by $u_t(\alpha_1)^{-1}$.) Let $v_1 = u_1$ and, for $t \in \{2, 3, \ldots, m\}$ let $v_t = u_t - u_t(\alpha_1)\, u_1$. Let $\alpha_d = \inf S(\langle v_2, \ldots, v_m \rangle)$. Then $d > 1$ since $v_t(\alpha_1) = 0$ for $t > 1$. We may assume that $v_2(\alpha_d) = 1$.

Let $w_2 = v_2$ and, for $t \in \{1, 3, 4, \ldots, m\}$, let $w_t = v_t - v_t(\alpha_d)\, v_2$. Then $U = \langle w_1, w_2, \ldots, w_m \rangle$ and $\langle v_2, \ldots, v_m \rangle = \langle w_2, \ldots, w_m \rangle$. Also if $1 \leqslant j < d$ then $w_1(\alpha_j) \neq 0$ (since $\alpha_j \in S(U) \backslash S(\langle v_2, \ldots, v_m \rangle)$) and $w_t(\alpha_j) = 0$ if $t \geqslant 2$. If $t \neq 2$ then $w_t(\alpha_d) = 0$.

From here the proof proceeds by two cases. Assume first that $q > 2$. Let $c \in F \backslash \{0, 1\}$, let $W_1 = \langle w_1 + w_2, w_3, \ldots, w_m \rangle$, and let $W_2 = \langle w_1 + cw_2, w_3, \ldots, w_m \rangle$. Then by our assumptions $\{W_1, W_2\} \subseteq \Gamma(\mathfrak{k}, U) \subseteq A_{\sigma,i}$. Thus in particular $W_1{}^* = W_2{}^*$ so that $OI(w_1 + w_2, W_1) \in W_2{}^*$. Thus there exists $\{a_t\}_{t=2}^{m} \subseteq F$ such that $OI(w_1 + w_2, W_1) = a_2 OI(w_1 + w_2, W_2) + \sum_{t=3}^{m} a_t OI(w_t, W_2)$. Note that $\alpha_1, \alpha_2, \ldots, \alpha_d$ are the first d elements of $S(W_1)$ and of $S(W_2)$. Consequent,ly $1 = (w_1 + w_2)(\alpha_1) = OI(w_1 + w_2, W_1)(1) = a_2 OI(w_1 + cw_2, W_2)(1) + \sum_{t=3}^{m} a_t OI(w_t, W_2)(1) = a_2 OI(w_1 + cw_2, W_2)(1) = a_2(w_1 + cw_2)(\alpha_1) = a_2 \cdot 1$ so that $a_2 = 1$. On the other hand $1 = (w_1 + w_2)(\alpha_d) = OI(w_1 + w_2, W_1)(d) = a_2 OI(w_1 + cw_2, W_2)(d) + \sum_{t=3}^{m} a_t OI(w_t, W_2)(d) = \alpha_2(w_1 + cw_2)(\alpha_d) = a_2 c$. Thus $a_2 = c^{-1}$, a contradiction.

For the second case we assume that $q = 2$ and $\mathfrak{k} > 1$. Let $W_3 = \langle w_1, w_2, \ldots, w_{m-1} \rangle$ and let $W_4 = \langle w_1 + w_2, \ldots, w_m \rangle$. Then, by our assumptions $\{W_3, W_4\} \subseteq \Gamma(\mathfrak{k}, U) \subseteq A_{\sigma,i}$ so that, in particular, $W_3{}^* = W_4{}^*$. Consequently $OI(w_1, W_3) \in W_4{}^*$.

Note that $\alpha_1, \alpha_2, \ldots, \alpha_d$ are the first d elements of both $S(W_3)$ and $S(W_4)$. (It is at this point that we use the assumption that $\mathfrak{k} > 1$, for otherwise α_d would not be in the support of W_3.) Now there exists $\{a_t\}_{t=2}^{m}$ such that $OI(w_1, W_3) = a_2 OI(w_1 + w_2, W_4) + \sum_{t=3}^{m} a_t OI(w_t, W_4)$. Thus $1 = w_1(\alpha_1) = OI(w_1, W_3)(1) = a_2 OI(w_1 + w_2, W_4)(1) = a_2(w_1 + w_2)(\alpha_1) = a_2$ so $a_2 = 1$. Also $0 = w_1(\alpha_d) = OI(w_1, W_3)(d) = a_2 OI(w_1 + w_2, W_4)(d) = a_2(w_1 + w_2)(\alpha_d) = a_2$, a contradiction.

The completion of the proof of Lemma 2.9 brings us to the one class of cases for which the validity of Ψ is not completely known. Specifically our remaining concern is with $\Psi(1, m, n, 2, r)$ where $m < \aleph_0$ and $r \geqslant \aleph_0$. It turns out that, for a given m and r, there are a finite number of values of n for which we do not know the answer.

2.10. Definition. Let G be a set of linearly independent vectors and let H be a nonempty subset of G.

(a) $D(H, G) = \bigcap \{S(u) : u \in H\} \backslash \bigcup \{S(u) : u \in G \backslash H\}$.

(b) $d(H, G) = |\, D(H, G)|$.

Note that if H and H' are distinct nonempty subsets of G then

$D(H, G) \cap D(H', G) = \varnothing$. (The reader may find it helpful to note that each $D(H, G)$ is one region of the Venn diagram of the supports of elements of G.)

2.11. LEMMA. *Let V be an $\mathfrak{n}$-dimensional vector space over $GF(2)$ and let $U \in \Gamma(\mathfrak{m}, V)$ (where $1 < \mathfrak{m} < \aleph_0$). Let U^* be any basis for U. The following statements are equivalent.*

(1) *If $\{W, W'\} \subseteq \Gamma(1, U)$ then $s(W) = s(W')$;*

(2) *If H and H' are nonempty subsets of U^* then $d(H, U^*) = d(H', U^*)$;*

(3) *If $W \in \Gamma(1, U)$ and H is a nonempty subset of U^* then $s(W) = 2^{\mathfrak{m}-1} \cdot d(H, U^*)$.*

Proof. Write $U^* = \{u_1, u_2, ..., u_{\mathfrak{m}}\}$.

(1) *implies* (2). The proof proceeds by induction on $\mathfrak{m}$. Suppose first $\mathfrak{m} = 2$. Let $H_1 = \{u_1\}$, $H_2 = \{u_2\}$, and $H_3 = \{u_1, u_2\}$. Then $s(\langle u_1\rangle) = s(\langle u_2\rangle)$ while $S(\langle u_1\rangle) = D(H_1, U^*) \cup D(H_3, U^*)$ and $S(\langle u_2\rangle) = D(H_2, U^*) \cup D(H_3, U^*)$. Thus $d(H_1, U^*) = d(H_2, U^*)$. Similarly, since $s(\langle u_1\rangle) = s(\langle u_1 + u_2\rangle)$ one has that $| D(H_1, U^*) \cup D(H_3, U^*)| = | D(H_1, U^*) \cup D(H_2, U^*)|$ so that $d(H_3, U^*) = d(H_2, U^*)$.

Now suppose $\mathfrak{m} > 2$. We show first that, if $H \subseteq U^*$, $| H | > 2$, and u_α and u_β are distinct elements of H, then $d(H, U^*) = d(H\backslash\{u_\alpha, u_\beta\}, U^*)$. Let $K = U^*\backslash\{u_\alpha\}$ and let $K' = U^*\backslash\{u_\beta\}$. Then by induction we have that $d(H\backslash\{u_\alpha\}, K) = d(H\backslash\{u_\alpha, u_\beta\}, K)$ and $d(H\backslash\{u_\beta\}, K') = d(H\backslash\{u_\alpha, u_\beta\}, K')$. But $D(H\backslash\{u_\alpha\}, K) = D(H\backslash\{u_\alpha\}, U^*) \cup D(H, U^*)$ and $D(H\backslash\{u_\alpha, u_\beta\}, K) = D(H\backslash\{u_\alpha, u_\beta\}, U^*) \cup D(H\backslash\{u_\beta\}, U^*)$. Also, $D(H\backslash\{u_\beta\}, K') = D(H\backslash\{u_\beta\}, U^*) \cup D(H, U^*)$ and $D(H\backslash\{u_\alpha, u_\beta\}, K') = D(H\backslash\{u_\alpha, u_\beta\}, U^*) \cup D(H\backslash\{u_\alpha\}, U^*)$. Consequently we have the two equations $d(H\backslash\{u_\alpha\}, U^*) + d(H, U^*) = d(H\backslash\{u_\alpha, u_\beta\}, U^*) + d(H\backslash\{u_\beta\}, U^*)$ and $d(H\backslash\{u_\beta\}, U^*) + d(H, U^*) = d(H\backslash\{u_\alpha, u_\beta\}, U^*) + d(H\backslash\{u_\alpha\}, U^*)$. From these we conclude that $d(H, U^*) = d(H\backslash\{u_\alpha, u_\beta\}, U^*)$ as desired.

Now, for each $\mathfrak{i}$ in $\{1, 2, ..., \mathfrak{m}\}$ let $H_{\mathfrak{i}} = U^*\backslash\{u_{\mathfrak{i}}\}$. If $\mathfrak{i} \neq \mathfrak{j}$, let $\mathfrak{h} \in \{1, 2, ..., \mathfrak{m}\}\backslash\{\mathfrak{i}, \mathfrak{j}\}$ and note that $H_{\mathfrak{i}}\backslash\{u_{\mathfrak{j}}, u_{\mathfrak{h}}\} = H_{\mathfrak{j}}\backslash\{u_{\mathfrak{i}}, u_{\mathfrak{h}}\}$ so that $d(H_{\mathfrak{i}}, U^*) = d(H_{\mathfrak{i}}\backslash\{u_{\mathfrak{j}}, u_{\mathfrak{h}}\}, U^*) = d(H_{\mathfrak{j}}\backslash\{u_{\mathfrak{i}}, u_{\mathfrak{h}}\}, U^*) = d(H_{\mathfrak{j}}, U^*)$.

Let $\mathfrak{a} = d(U^*, U^*)$ and let $\mathfrak{b} = d(H_1, U^*)$. Let H be a nonempty subset of U^*. If $| H | = \mathfrak{m} \pmod 2$ then, since $H \subseteq U^*$, one has $d(H, U^*) = \mathfrak{a}$. If $| H | \neq \mathfrak{m} \pmod 2$ then $H \subseteq H_{\mathfrak{i}}$ for some $\mathfrak{i}$ and $| H | = | H_{\mathfrak{i}} | \pmod 2$ so $d(H, U^*) = \mathfrak{b}$. (In each case one simply throws away elements two at a time to reach H.) Thus to complete the proof it suffices to show $\mathfrak{a} = \mathfrak{b}$.

To this end note that $S(\langle u_1\rangle) = \bigcup \{D(H, U^*)\colon u_1 \in H$ and $| H | = \mathfrak{m} \pmod 2\} \cup \bigcup \{D(H, U^*)\colon u_1 \in H$ and $| H | \neq \mathfrak{m} \pmod 2\}$. Thus $s(\langle u_1\rangle) =$

$2^{\mathfrak{m}-2}\mathfrak{a} + 2^{\mathfrak{m}-2}\mathfrak{b}$. Also $S(\langle u_1 + u_2 + \cdots + u_{\mathfrak{m}}\rangle) = \bigcup \{D(H, U^*): |H| = 1 \pmod 2\}$, so that $s(\langle u_1 + u_2 + \cdots + u_{\mathfrak{m}}\rangle) = 2^{\mathfrak{m}-1}\mathfrak{c}$ where $\mathfrak{c} = \mathfrak{a}$ if $\mathfrak{m}$ is odd and $\mathfrak{c} = \mathfrak{b}$ if $\mathfrak{m}$ is even. In either case, since $s(\langle u_1\rangle) = s(\langle u_1 + u_2 + \cdots + u_{\mathfrak{m}}\rangle)$, one has $\mathfrak{a} = \mathfrak{b}$ as desired.

(2) *implies* (3). Let $\mathfrak{a} = d(U^*, U^*)$. By assumption, if H is any nonempty subset of U^*, then $d(H, U^*) = \mathfrak{a}$. Let $W \in \Gamma(1, U)$. Then $W = \langle w\rangle$ where $w = \sum H_0$ for some nonempty subset H_0 of U^*. Now $S(W) = \bigcup \{D(H, U^*): |H \cap H_0| = 1 \pmod 2\}$ and $|\{H \subseteq U^*: |H \cap H_0| = 1 \pmod 2\}| = 2^{\mathfrak{m}-1}$ so that $s(W) = 2^{\mathfrak{m}-1}\mathfrak{a}$ as desired.

(3) *implies* (1). Trivial.

The validity of the following lemma is established by an easy induction.

2.12. Lemma. *Let $0 < \mathfrak{m} < \aleph_0$, let $\mathfrak{n} \geqslant \aleph_0$, and let V be an $\mathfrak{n}$-dimensional vector space over $GF(2)$. Let $U \in \Gamma(\mathfrak{m}, V)$ and let $\alpha \in S(U)$. Then $|\{u \in U: \alpha \in S(u)\}| = 2^{\mathfrak{m}-1}$. Further, if $\{\beta_1, \beta_2, \ldots, \beta_{\mathfrak{t}}\} \subseteq S(U)$ and, for each $\mathfrak{i}$, $U_{\mathfrak{i}} = \{u \in U: \beta_i \notin S(u)\}$ then $|\bigcap_{\mathfrak{i}=1}^{\mathfrak{t}} U_{\mathfrak{i}}| = 2^{\mathfrak{m}-\mathfrak{j}}$ for some $\mathfrak{j} \leqslant \mathfrak{t}$.*

2.13. Lemma. *Let $0 < \mathfrak{m} < \aleph_0$, let $\mathfrak{n} \geqslant \aleph_0$, and let V be an $\mathfrak{n}$-dimensional vector space over $GF(2)$. Let $U \in \Gamma(\mathfrak{m}, V)$ and let $\{\beta_1, \beta_2, \ldots, \beta_{\mathfrak{t}}\} \subseteq S(U)$. If $w \in U$ such that $\{\beta_1, \beta_2, \ldots, \beta_{\mathfrak{t}}\} \subseteq S(w)$ then $|\{u \in U: \{\beta_1, \beta_2, \ldots, \beta_{\mathfrak{t}}\} \subseteq S(u)\}| \geqslant 2^{\mathfrak{m}-\mathfrak{t}}$.*

Proof. Let $U_{\mathfrak{i}}$ be as in Lemma 2.12. Then $\{u \in U: \{\beta_1, \beta_2, \ldots, \beta_{\mathfrak{t}}\} \subseteq S(u)\} = U \backslash \bigcup_{\mathfrak{i}=1}^{\mathfrak{t}} U_{\mathfrak{i}}$. But the cardinality of $\bigcup_{\mathfrak{i}=1}^{\mathfrak{t}} U_{\mathfrak{i}}$ is, by a familiar counting formula, made up of sums and differences of the cardinalities of finite intersections of $\{U_{\mathfrak{i}}\}_{\mathfrak{i}=1}^{\mathfrak{t}}$. Each of these finite intersections has, by Lemma 2.12, a multiple of $2^{\mathfrak{m}-\mathfrak{t}}$ elements so $|\bigcup_{\mathfrak{i}=1}^{\mathfrak{t}} U_{\mathfrak{i}}|$ is a multiple of $2^{m-\mathfrak{t}}$. By hypothesis $\bigcup_{\mathfrak{i}=1}^{\mathfrak{t}} U_{\mathfrak{i}} \neq U$, so $U \backslash \bigcup_{\mathfrak{i}=1}^{\mathfrak{t}} U_{\mathfrak{i}}$ has at least $2^{\mathfrak{m}-\mathfrak{t}}$ elements.

Recall that for a cardinal $\mathfrak{r} \geqslant \aleph_0$, $\mathfrak{r} = \aleph_\alpha$ where α is the order type of the set of infinite cardinals preceeding $\mathfrak{r}$.

2.14. Lemma. *Let $\mathfrak{r} = \aleph_\alpha$ and let $1 < \mathfrak{m} < \aleph_0$. Let $\mathfrak{m} \leqslant \mathfrak{n} < \aleph_{\alpha+m}$. Then $\Psi(1, \mathfrak{m}, \mathfrak{n}, 2, \mathfrak{r})$ fails.*

Proof. We may assume that $\mathfrak{n} = \aleph_{\alpha+\mathfrak{m}-1}$. Let V be an $\mathfrak{n}$-dimensional vector space over $GF(2)$. For each $\sigma < \mathfrak{m}$ let $A_\sigma = \{W \in \Gamma(1, V): s(W) = \sigma\}$.

For each subset H of $\aleph_{\alpha+\mathfrak{m}-1}$, let $R(H)$ be a well ordering of H of order type $|H|$. If $|H| \leqslant \aleph_\alpha$ let g_H be a map taking the finite subsets of H one-to-one into $\aleph_\alpha \backslash \{0, 1, \ldots, \mathfrak{m} - 1\}$.

For each W in $\Gamma(1, V)$ with $s(W) \geqslant \mathfrak{m}$ choose $\{\beta(1, W), \ldots, \beta(\mathfrak{m} - 1, W)\} \subseteq S(W)$ and $\{H(1, W), \ldots, H(m - 1, W)\}$ as follows. Let $\beta(1, W) = \sup S(W)$ and let $H(1, W) = \{\gamma: \gamma < \beta(1, W)\}$. Note that $|H(1, W)| \leqslant \aleph_{\alpha+\mathfrak{m}-2}$ and that $S(W) \backslash \{\beta(1, W)\} \subseteq H(1, W)$.

Let $\beta(2, W)$ be the sup of $S(W)\backslash\{\beta(1, W)\}$ under the order $R(H(1, W))$ and let $H(2, W) = \{\gamma: \gamma$ precedes $\beta(2, W)$ under the order $R(H(1, W))\}$. Continuing in this fashion we choose $\{\beta(1, W),..., \beta(\mathfrak{m} - 1, W)\}$ and $\{H(1, W),..., H(\mathfrak{m} - 1, W)\}$ and note that $| H(\mathfrak{m} - 1, W)| \leqslant \aleph_\alpha$. Letting $G(W) = S(W)\backslash\{\beta(1, W),..., \beta(\mathfrak{m} - 1, W)\}$ we also note that $G(W) \subseteq H(\mathfrak{m} - 1, W)$.

Now for each σ such that $\mathfrak{m} \leqslant \sigma < \aleph_\alpha$, let $A_\sigma = \{W \in \Gamma(1, V)$: $g_{H(\mathfrak{m}-1,W)}(G(W)) = \sigma\}$. Suppose that there are some U in $\Gamma(\mathfrak{m}, V)$ and some $\sigma < \aleph_\alpha$ such that $\Gamma(1, U) \subseteq A_\sigma$. Note that $\sigma \geqslant \mathfrak{m}$. For if $\sigma < \mathfrak{m}$ then each W in $\Gamma(1, U)$ would have $s(W) = \sigma$. But then, by Lemma 2.11, σ would be a multiple of $2^{\mathfrak{m}-1}$, a contradiction. Consequently, whenever $W \in \Gamma(1, U)$ one has $s(W) \geqslant \mathfrak{m}$.

Now let $\beta_1 = \sup S(U)$, let $F_1 = \{u \in U: \beta_1 \in S(u)\}$, and let $H_1 = \{\gamma: \gamma < \beta_1\}$. Note that $F_1 \neq \varnothing$ so, by Lemma 2.13, $| F_1 | \geqslant 2^{\mathfrak{m}-1}$ (in fact $| F_1 | = 2^{\mathfrak{m}-1}$). Note also that $S(F_1)\backslash\{\beta_1\} \subseteq H_1$. Let β_2 be the sup of $S(F_1)\backslash\{\beta_1\}$ under the order $R(H_1)$, let $F_2 = \{u \in U: \{\beta_1, \beta_2\} \subseteq S(u)\}$, and let $H_2 = \{\gamma: \gamma$ precedes β_2 under $R(H_1)\}$. Note that $| F_2 | \geqslant 2^{\mathfrak{m}-2}$ and that $S(F_2)\backslash\{\beta_1, \beta_2\} \subseteq H_2$. Continuing in this fashion we obtain $\{\beta_1,..., \beta_{\mathfrak{m}-1}\}$, $\{F_1,..., F_{\mathfrak{m}-1}\}$, and $\{H_1,..., H_{\mathfrak{m}-1}\}$. (At each stage we could choose $\beta_{\mathfrak{i}}$ since each member u of $U\backslash\{0\}$ had $s(u) \geqslant \mathfrak{m}$.) The method of choice of $\beta_{\mathfrak{i}}$ ensured that $F_{\mathfrak{m}-1} \neq \varnothing$ so $| F_{\mathfrak{m}-1} | \geqslant 2$. Let u and u' be distinct members of $F_{\mathfrak{m}-1}$, let $W = \langle u \rangle$, and let $W' = \langle u' \rangle$. Then for each $\mathfrak{i}$ in $\{1, 2,..., \mathfrak{m} - 1\}$ we have $\beta_{\mathfrak{i}} = \beta(\mathfrak{i}, W) = \beta(\mathfrak{i}, W')$ and $H_{\mathfrak{i}} = H(\mathfrak{i}, W) = H(\mathfrak{i}, W')$. Consequently $g_{H(\mathfrak{m}-1,W)} = g_{H(\mathfrak{m}-1,W')}$. But $W \neq W'$ so $G(W) \neq G(W')$ so that $g_{H(\mathfrak{m}-1,W)}(G(W)) \neq g_{H(\mathfrak{m}-1,W')}(G(W'))$, a contradiction.

The final result needed for the proof of the main theorem is the only new result for which $\Psi(\mathfrak{k}, \mathfrak{m}, \mathfrak{n}, \mathfrak{q}, \mathfrak{r})$ holds. It is an easy consequence of the following result of Erdös and Rado [1] and of Kurepa [5], which we state here as Lemma 2.15. We shall use the notation $\mathscr{P}_{\mathfrak{t}}(X)$ for the set of $\mathfrak{t}$ element subsets of X.

2.15. Lemma (Erdös, Kurepa, Rado). *Assume the generalized continuum hypothesis. If* $\mathscr{P}_{\mathfrak{t}}(\aleph_{\alpha+t}) = \bigcup_{\sigma<\aleph_\alpha} B_\sigma$ *then there exist* $\tau < \aleph_\alpha$ *and* $C \subseteq \aleph_{\alpha+\mathfrak{t}}$ *such that* $| C | = \aleph_{\alpha+1}$ *and* $\mathscr{P}_{\mathfrak{t}}(C) \subseteq B_\tau$. *(That is,* $\aleph_{\alpha+\mathfrak{t}} \to (\aleph_{\alpha+1})^{\mathfrak{t}}_{\aleph_\alpha}$.*)*

2.16. Lemma. *Assume the generalized continuum hypothesis. Let* $\mathfrak{r} = \aleph_\alpha$, *let* $1 < \mathfrak{m} < \aleph_0$, *and let* $\mathfrak{n} \geqslant \aleph_{\alpha+2^{\mathfrak{m}-1}}$. *Then* $\Psi(1, \mathfrak{m}, \mathfrak{n}, 2, \mathfrak{r})$ *holds.*

Proof. Let V be an $\mathfrak{n}$-dimensional vector space over $GF(2)$. We may assume $\mathfrak{n} = \aleph_{\alpha+2^{\mathfrak{m}-1}}$. Let $\Gamma(1, V) = \bigcup_{\sigma<\mathfrak{r}} A_\sigma$ and, for each $\sigma < \mathfrak{r}$, let $B_\sigma = \{S(W): W \in A_\sigma$ and $s(W) = 2^{\mathfrak{m}-1}\}$. Let $\mathfrak{t} = 2^{\mathfrak{m}-1}$. Then $\mathscr{P}_{\mathfrak{t}}(\mathfrak{n}) = \bigcup_{\sigma<\mathfrak{r}} B_\sigma$ so there exist $\tau < \mathfrak{r}$ and $C \subseteq \mathfrak{n}$ such that $| C | = \aleph_{\alpha+1}$ and $\mathscr{P}_{\mathfrak{t}}(C) \subseteq B_\tau$. (In fact all we need here is that $| C | \geqslant 2^{\mathfrak{m}} - 1$.)

Let $\beta_1, \beta_2, ..., \beta_{2^{\mathfrak{m}}-1}$ be distinct members of C and let $\{H_1, ..., H_{2^{\mathfrak{m}}-1}\}$ be the set of nonempty subsets of $\{1, 2, ..., \mathfrak{m}\}$. For $\mathfrak{j}$ in $\{1, 2, ..., \mathfrak{m}\}$ let $u_{\mathfrak{j}}(\beta_{\mathfrak{i}}) = 1$ if $\mathfrak{j} \in H_{\mathfrak{i}}$ and let $u_{\mathfrak{j}}(\delta) = 0$ otherwise. Let $U = \langle u_1, ..., u_{\mathfrak{m}}\rangle$. Then $S(U) = \{\beta_1, \beta_2, ..., \beta_{2^{\mathfrak{m}}-1}\} \subseteq C$. Let $U^* = \{u_1, ..., u_{\mathfrak{m}}\}$. If H is a nonempty subset of U^* then $H = \{u_{\mathfrak{j}}: \mathfrak{j} \in H_{\mathfrak{i}}\}$ for some $\mathfrak{i}$. Thus $D(H, U^*) = \{\beta_{\mathfrak{i}}\}$. Thus, by Lemma 2.11, $s(W) = 2^{\mathfrak{m}-1}$ whenever $W \in \Gamma(1, U)$. But then $\{S(W): W \in \Gamma(1, U)\} \subseteq B_\tau$ so that $\Gamma(1, U) \subseteq A_\tau$.

3. The Partition Theorems

As previously remarked, we are assuming that the conclusions about $\Psi(\mathfrak{k}, \mathfrak{m}, \mathfrak{n}, \mathfrak{q}, \mathfrak{r})$ are not vacuous so that $\mathfrak{k} \leqslant \mathfrak{m} \leqslant \mathfrak{n}$, $\mathfrak{r} > 0$, and $\mathfrak{q}$ is either infinite or the positive power of a prime.

3.1. Theorem. *Let* $\mathfrak{r} < \aleph_0$. *Then* $\Psi(\mathfrak{k}, \mathfrak{m}, \mathfrak{n}, \mathfrak{q}, \mathfrak{r})$ *holds if and only if one of the following holds.*

(1) $\mathfrak{r} = 1$;

(2) $\mathfrak{k} = \mathfrak{m} < \aleph_0$;

(3) $\mathfrak{k} = 0$;

(4) $\mathfrak{k} < \aleph_0$, $\mathfrak{m} < \aleph_0$, $\mathfrak{q} < \aleph_0$, *and* $\mathfrak{n} > N(\mathfrak{k}, \mathfrak{m}, \mathfrak{q}, \mathfrak{r})$; *or*

(5) $\mathfrak{k} = 1$, $\mathfrak{q} = 2$, *and* $\mathfrak{m} = \aleph_0$.

Proof. Sufficiency. The sufficiency of each of (1), (2), and (3) is trivial. The sufficiency of (4) is Theorem 1.2 [2, Corollary 2] and the sufficiency of (5) is [4, Corollary 3.5].

Necessity. Suppose that each of statements (1)–(5) is false. Then in particular (1) and (3) fail so we know that $\mathfrak{k} \geqslant 1$ and $\mathfrak{r} \geqslant 2$. Using these two facts we may write the negations of (2), (4), and (5) as follows.

Not (2): $1 \leqslant \mathfrak{k} < \mathfrak{m} < \aleph_0$ or $\mathfrak{m} \geqslant \aleph_0$.

Not (4): $\mathfrak{k} \geqslant \aleph_0$ or ($\mathfrak{k} < \aleph_0$ and $\mathfrak{m} \geqslant \aleph_0$) or ($\mathfrak{k} < \aleph_0$ and $\mathfrak{m} < \aleph_0$ and $\mathfrak{q} \geqslant \aleph_0$) or ($\mathfrak{k} < \aleph_0$ and $\mathfrak{m} < \aleph_0$ and $\mathfrak{q} < \aleph_0$ and $\mathfrak{n} \leqslant N(\mathfrak{k}, \mathfrak{m}, \mathfrak{q}, \mathfrak{r})$).

Not (5): $\mathfrak{k} > 1$ or ($\mathfrak{k} = 1$ and $\mathfrak{q} > 2$) or ($\mathfrak{k} = 1$ and $\mathfrak{q} = 2$ and $\mathfrak{m} < \aleph_0$) or ($\mathfrak{k} = 1$ and $\mathfrak{q} = 2$ and $\mathfrak{m} > \aleph_0$).

Using these one concludes that one of the following must hold.

(a) $\mathfrak{m} \geqslant \aleph_0$ and $\mathfrak{k} \geqslant 2$;

(b) $\mathfrak{k} = 1$ and $\mathfrak{m} \geqslant \aleph_0$ and $\mathfrak{q} > 2$;

(c) $\mathfrak{k} = 1$ and $\mathfrak{m} > \aleph_0$ and $\mathfrak{q} = 2$;

(d) $0 < \mathfrak{k} < \mathfrak{m} < \aleph_0$ and $\mathfrak{q} \geqslant \aleph_0$; or

(e) $0 < \mathfrak{k} < \mathfrak{m} < \aleph_0$ and $\mathfrak{q} < \aleph_0$ and $\mathfrak{n} \leqslant N(\mathfrak{k}, \mathfrak{m}, \mathfrak{q}, \mathfrak{r})$.

But these cannot hold by Lemmas 2.3, 2.4, 2.5, 2.7, and Theorem 1.2 respectively.

The appeal to the generalized continuum hypothesis in the following theorem applies only to the sufficiency of condition (6).

3.2. THEOREM. *Assume the generalized continuum hypothesis and exclude the possibility that simultaneously* $\mathfrak{r} = \aleph_\alpha$, $\mathfrak{k} = 1$, $\mathfrak{m} < \aleph_0$, $\mathfrak{q} = 2$, *and* $\aleph_{\alpha+\mathfrak{m}} \leqslant \mathfrak{n} < \aleph_{\alpha+2^{\mathfrak{m}-1}}$. *Then* $\Psi(\mathfrak{k}, \mathfrak{m}, \mathfrak{n}, \mathfrak{q}, \mathfrak{r})$ *holds if and only if one of the following holds.*

(1) $\mathfrak{r} = 1$;

(2) $\mathfrak{k} = \mathfrak{m} < \aleph_0$;

(3) $\mathfrak{k} = 0$;

(4) $\mathfrak{k} < \aleph_0$, $\mathfrak{m} < \aleph_0$, $\mathfrak{q} < \aleph_0$, $\mathfrak{r} < \aleph_0$, *and* $\mathfrak{n} > N(\mathfrak{k}, \mathfrak{m}, \mathfrak{q}, \mathfrak{r})$;

(5) $\mathfrak{k} = 1$, $\mathfrak{q} = 2$, $\mathfrak{r} < \aleph_0$, *and* $\mathfrak{m} = \aleph_0$; *or*

(6) $\mathfrak{k} = 1$, $\mathfrak{q} = 2$, $\mathfrak{m} < \aleph_0$, $\mathfrak{r} = \aleph_\alpha$, *and* $\mathfrak{n} \geqslant \aleph_{\alpha+2^{\mathfrak{m}-1}}$.

Proof. Sufficiency. The sufficiency of each of (1), (2), and (3) is trivial. The sufficiency of each of (4) and (5) follows from Theorem 2.1. The sufficiency of (6) is Lemma 2.16.

Necessity. If each of (1)–(5) fails we conclude that $\mathfrak{r} \geqslant \aleph_0$ from Theorem 2.1.

If in addition (6) fails we have that $\mathfrak{k} > 1$ or ($\mathfrak{k} = 1$ and $\mathfrak{q} > 2$) or ($\mathfrak{k} = 1$ and $\mathfrak{q} = 2$ and $\mathfrak{m} \geqslant \aleph_0$) or ($\mathfrak{k} = 1$ and $\mathfrak{q} = 2$ and $\mathfrak{m} < \aleph_0$ and $\mathfrak{n} < \aleph_{\alpha+2}\mathfrak{m} - 1$). The first two cases are impossible by Lemma 2.9. The third case is impossible by Lemma 2.8. If the fourth case holds then by the exclusion in the hypothesis $\mathfrak{n} < \aleph_{\alpha+\mathfrak{m}}$, which is impossible by Lemma 2.14.

Theorem 2.2 tells us that, given cardinals $\mathfrak{k}$, $\mathfrak{m}$, $\mathfrak{q}$, and $\mathfrak{r}$, we know the validity of $\Psi(\mathfrak{k}, \mathfrak{m}, \mathfrak{n}, \mathfrak{q}, \mathfrak{r})$ for all except a finite number of values for $\mathfrak{n}$.

4. QUESTIONS AND REMARKS

We immediately state the obvious question.

4.1. QUESTION. Let $\mathfrak{r} = \aleph_\alpha$, $2 < \mathfrak{m} < \aleph_0$, and $\aleph_{\alpha+\mathfrak{m}} \leqslant \mathfrak{n} < \aleph_{\alpha+2}\mathfrak{m} - 1$. Does $\Psi(1, \mathfrak{m}, \mathfrak{n}, 2, \mathfrak{r})$ hold?

While $N(\mathfrak{k}, \mathfrak{m}, \mathfrak{q}, \mathfrak{r})$ is computable with $\mathfrak{k}$, $\mathfrak{m}$, $\mathfrak{q}$, and $\mathfrak{r}$ finite, it is known for relatively few values of $\mathfrak{k}$, $\mathfrak{m}$, $\mathfrak{q}$ and $\mathfrak{r}$.

4.2. QUESTION. Let $\mathfrak{k}$, $\mathfrak{m}$, $\mathfrak{q}$, and $\mathfrak{r}$ be finite. What is $N(\mathfrak{k}, \mathfrak{m}, \mathfrak{q}, \mathfrak{r})$?

It should be observed that the affirmative answers to the question of the validity of $\Psi(\mathfrak{k}, \mathfrak{m}, \mathfrak{n}, \mathfrak{q}, \mathfrak{r})$ are all either

(1) trivial (conditions (1)–(3) of Theorem 2.2),

(2) set theoretic (conditions (5) and (6) of Theorem 2.2) where the requirement that $\mathfrak{k} = 1$ and $\mathfrak{q} = 2$ transforms the statement into one about unions and intersections of sets); or (3) Rota's conjecture. The point of this remark is that Rota's conjecture (proved by Graham, Leeb, and Rothschild) includes all nontrivial cases which significantly involve the properties of a vector space.

It should be noted finally that the authors' original motivation for the study of this problem arose from an attempt to generalize the results of Graham and Rothschild [3] on $\mathfrak{n}$-parameter sets to $\aleph_0$-parameter sets, appropriately defined. The result of that work was that only in certain very restricted and essentially trivial cases does one obtain a partition theorem for $\aleph_0$-parameter sets. The necessary counterexamples can be obtained using the methods of Section 2 of this paper.

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